

The average rate of change = $\frac{\Delta y}{\Delta t}$.

When Limit $\Delta t \rightarrow 0$ is applied, the rate of change becomes instantaneous and we get the rate of change of y w.r.t. time at an instant.

i.e., $\lim_{\Delta t \rightarrow 0} \frac{\Delta y}{\Delta t} = \frac{dy}{dt}$.

$\left(\frac{dy}{dx}\right)_p = \tan \theta = \text{slope of tangent at } P.$

Equation of Tangent and Normal

Tangent at (x_1, y_1) is given by $(y - y_1) = f'(x_1)(x - x_1)$; when, $f'(x_1)$ is real.

And normal at (x_1, y_1) is $(y - y_1) = -\frac{1}{f'(x_1)}(x - x_1)$, when $f'(x_1)$ is nonzero real.

Note:

1. If tangent is parallel to x -axis, $\theta = 0^\circ \Rightarrow \tan \theta = 0$

$\therefore \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 0$

2. If tangent is perpendicular to x -axis (or parallel to y -axis) then $\theta = 90^\circ \Rightarrow \tan \theta \rightarrow \infty$ or $\cot \theta = 0$

$\therefore \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \infty$

Equation of tangent and normal in parametric form

Let the equation of the curve be expressed in the parametric form $x = g(t)$ and $y = \phi(t)$ where t is the parameter.

The equation of the tangent at a point $P(t)$,

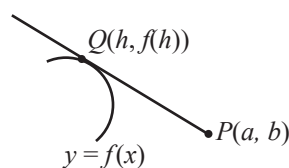
$y - \phi(t) = \frac{\phi'(t)}{g'(t)}[x - g(t)]$ and

the equation of normal is $y - \phi(t) = \frac{-g'(t)}{\phi'(t)}[x - g(t)]$

Tangent from an External Point

Given a point $P(a, b)$ which does not lie on the curve $y = f(x)$, then the equation of possible tangents to the curve passing through (a, b) can be found by solving for the point of contact Q .

$f'(h) = \frac{f(h) - b}{h - a}$



And equation of tangent is $y - b = \frac{f(h) - b}{h - a}(x - a)$

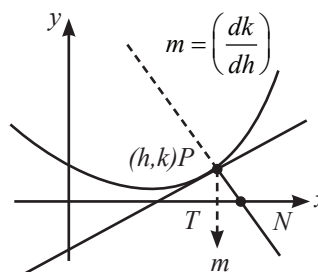
Length of Tangent, Normal, Subtangent, Subnormal at P(h,k)

1. $PT = |k| \sqrt{1 + \frac{1}{m^2}} = \text{Length of Tangent}$

2. $PN = |k| \sqrt{1 + m^2} = \text{Length of Normal}$

3. $TM = \left|\frac{k}{m}\right| = \text{Length of subtangent}$

4. $MN = |km| = \text{Length of subnormal.}$



Angle Between the Curves

Angle between two intersecting curves is defined as the acute angle between their tangents (or normals) at the point of intersection of two curves.

$\tan \theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$

If $\theta = \pi/2$, then the two curves are said to cut each other orthogonally and the condition for this to happen is:

$$m_1 \times m_2 = -1 \Rightarrow f'(x_0) \times g'(x_0) = -1$$

Shortest Distance between two Curves

Shortest distance between two non-intersecting differentiable curves is always along their common normal. (Wherever defined)

Errors and Approximations

1. Errors: Let $y = f(x)$

From definition of derivative, $\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$

$$\frac{\Delta y}{\Delta x} = \frac{dy}{dx} \text{ approximately or } \Delta y = \left(\frac{dy}{dx} \right) \Delta x \text{ approximately}$$

Definition:

(i) Δx is known as **absolute error** in x .

(ii) $\frac{\Delta x}{x}$ is known as **relative error** in x .

(iii) $\frac{\Delta x}{x} \times 100\%$ is known as **percentage error** in x .

2. Approximations: From definition of derivative,

As Derivative of $f(x)$ at $(x = a) = f'(a)$

$$\text{or } f'(a) = \lim_{\delta x \rightarrow 0} \frac{f(a + \delta x) - f(a)}{\delta x}$$

$$\text{or } \frac{f(a + \delta x) - f(a)}{\delta x} \rightarrow f'(a) \quad (\text{approximately})$$

$$f(a + \Delta x) - f(a) \rightarrow \Delta x f'(a) \quad (\text{approximately}).$$

Properties of Monotonic Functions

1. If $f(x)$ is strictly increasing function on an interval $[a, b]$, then f^{-1} exists and it is also a strictly increasing function.
2. If $f(x)$ is strictly increasing function on an interval $[a, b]$ such that it is continuous, then f^{-1} is continuous on $[f(a), f(b)]$.
3. If $f(x)$ and $g(x)$ both are monotonically (or strictly) increasing (or decreasing) functions on $[a, b]$, then $g \circ f(x)$ is a monotonically (or strictly) increasing (in either case) function on $[a, b]$.
4. If one of the two functions $f(x)$ and $g(x)$ is strictly (or monotonically) increasing and other a strictly (monotonically) decreasing, then $g \circ f(x)$ is strictly (monotonically) decreasing (in either case) on $[a, b]$.
5. If $f(x)$ is increasing function then $\frac{1}{f(x)}$ is decreasing function for $f(x) \neq 0$.
6. If a function is invertible it has to be either increasing or decreasing.

Rolle's Theorem

If a function f defined on $[a, b]$ is

1. Continuous on $[a, b]$
2. derivable on (a, b) and
3. $f(a) = f(b)$.

Then there exists atleast one c ($a < c < b$) such that $f'(c) = 0$.

Lagrange's Mean Value Theorem (LMVT)

If a function f defined on $[a, b]$ is

1. continuous on $[a, b]$ is
2. derivable on (a, b)
3. $f(a) = f(b)$,

then there exists at least one real numbers between a and b ($a < c < b$) such

$$\text{that } \frac{f(b) - f(a)}{b - a} = f'(c).$$

Special Points

1. **Critical points:** The points of domain for which $f'(x)$ is equal to zero or doesn't exist are called critical points.
2. **Stationary points:** The stationary points are the points of domain where $f'(x) = 0$.

Note: Every stationary point is a critical point but vice-versa is not true.

Significance of the Sign of 2nd order Derivative and Point of Inflection

If $f''(x) > 0 \forall x \in (a, b)$ then graph of $f(x)$ is concave upward in (a, b) . Similarly if $f''(x) < 0 \forall x \in (a, b)$ then graph of $f(x)$ is concave downward in (a, b) .

Useful Formulae of Mensuration to Remember

1. Volume of a cuboid = ℓbh .
2. Surface area of cuboid = $2(\ell b + bh + h\ell)$.
3. Volume of cube = a^3 .
4. Surface area of cube = $6a^2$.
5. Volume of a cone = $\frac{1}{3}\pi r^2 h$.
6. Curved surface area of cone = $\pi r\ell$ (ℓ = slant height).
7. Curved surface area of a cylinder = $2\pi rh$.
8. Total surface area of a cylinder = $2\pi rh + 2\pi r^2$.
9. Volume of a sphere = $\frac{4}{3}\pi r^3$.
10. Surface area of a sphere = $4\pi r^2$.
11. Area of a circular sector = $\frac{1}{2}r^2\theta$, when θ is in radians.
12. Volume of a prism = (area of the base) \times (height).
13. Lateral surface area of a prism = (perimeter of the base) \times (height).
14. Total surface area of a prism = (lateral surface area) + 2 (area of the base).
(Note that lateral surfaces of a prism are all rectangle.)
15. Volume of a pyramid = $\frac{1}{3}$ (area of the base) \times (height).
16. Curved surface area of a pyramid = $\frac{1}{2}$ (perimeter of the base) \times (slant height).
(Note that slant surfaces of a pyramid are triangles).